

BIRATIONAL GEOMETRY OF THREEFOLDS VERSUS BIRATIONAL GEOMETRY OF FOLIATION

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Onions of Geometry



Fig. 1: Shrek realised it first

Foliations are ornaments for varieties (or manifolds) that make it look like an onion. In more technical terms:-

Let X be a normal quasi-projective variety (we consider everything over $\mathbb{C}$). A *foliation* $\mathcal{F}$ on X is a coherent subsheaf $\mathcal{F} \subset T_X$ of the tangent sheaf which is closed under Lie brackets and such that $T_X/\mathcal{F}$ is torsion free.

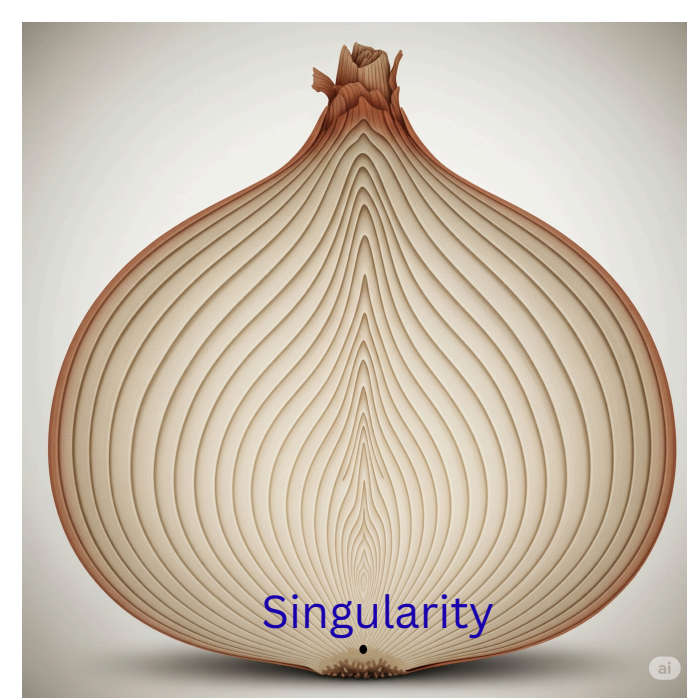


Fig. 2: Onions explain more math than you think

The main two toy examples to keep in mind

1. The foliation on $\mathbb{A}^2$ having leaves $xy = t$ (this is generated by $xdy + ydx$, cf. Fig. 3).
2. The linear projection from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ (this is generated by $xdy - ydx$, seen as a foliation on A^2 , see Fig.4).

As these structures are natural and provide non-trivial insights about the geometry of the underlying variety, it is important that we try to understand the birational geometry of these foliated structure.

Preparation for Cooking, Take out your Knives and Chopping Board

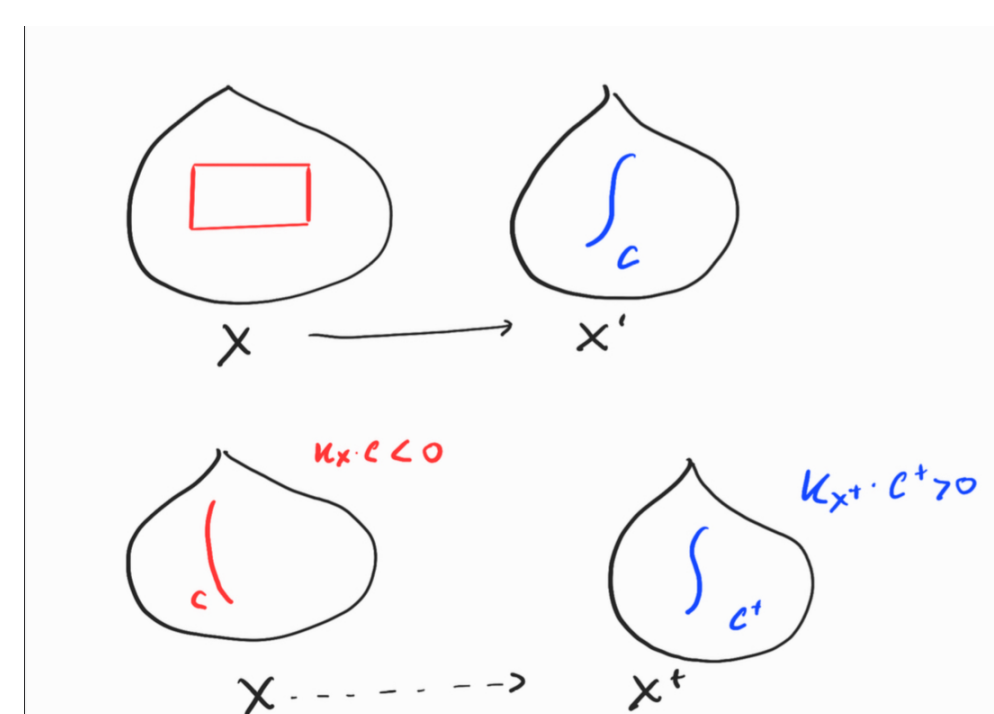


Fig. 5: Classical MMP/Traditional Chopping

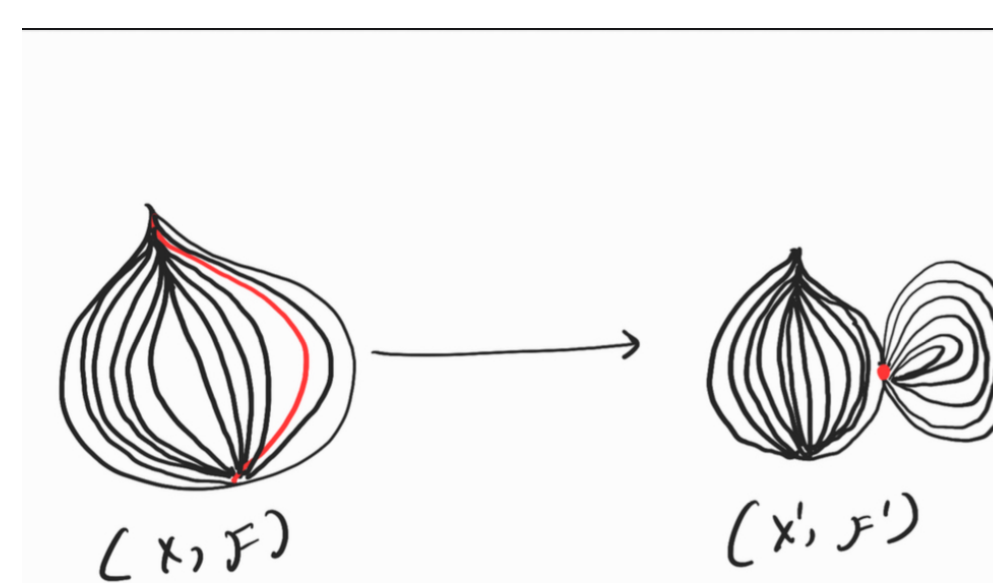


Fig. 6: Foliated MMP/ New Gen Chopping

Theorem 1 ([3],[4],[8]):-The MMP for foliations (rank 1 and 2) with canonical singularity on threefolds exist and any such mmp terminates.

The Fish and Chips case (Similarities)

In Fish and Chips, it does not really matter if you add onion rings to your chips or not. Similarly some features of birational geometry of algebraic varieties remains same (or similar at least) irrespective of looking at it via foliated lens.

Classical Threefold	Foliated Threefold
<i>chopping and replacing and outcome</i>	

In case of varieties there are two kind of end results,

1. Minimal Models, i.e. K_X is nef
2. Mori fiber spaces, i.e. a fibration $X \rightarrow Y$ with $-K_X$ being relatively ample.

If we foliate X with $\mathcal{F}$ (having mild singularities), we also get similar end results in dimension 3,

1. Foliated Minimal models, i.e. $K_{\mathcal{F}}$ is nef
2. Foliated Mori fiber spaces, i.e. a fibration $X' \rightarrow Y'$ tangent to $\mathcal{F}'$ with $-K_{\mathcal{F}'}$ being relatively ample

If X is a three-dimensional variety with at worst log canonical singularities, MMP exists and terminates.

Theorem 2([5])

MMP for co-rank one lc foliation $\mathcal{F}$ on klt threefold X , exists and terminates.

Relation between different outcomes

1. Starting from X if K_X is pseudo-effective, then we reach minimal models, and they can be transformed in to each other via special transformations called K_X -flops (due to Kawamata).

Theorem 3([5])

1. Starting from $(X, \mathcal{F})$ (having nice singularities), with co-rank of $\mathcal{F} = 1$ and $\dim X = 3$, and $K_{\mathcal{F}}$ pseudo-effective, then the foliated MMP terminates with foliated minimal model, and they are connected by $K_{\mathcal{F}}$ -flops.

2. Starting from X if we reach Mori fiber spaces, they can be transformed into each other by composition of 4 kinds of links, called Sarkisov links (c.f. [6],[2])

Theorem 4([7])

2. Let X be a normal quasi-projective threefold with a co-rank one foliation $\mathcal{F}$ on X having appropriate singularities, then the foliated Mori fiber spaces appearing as the end-product of foliated MMPs are connected by foliated Sarkisov links.

References

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Chicken Dopiyaza case (Differences)

Without onion (or *Piyaz*) it is not possible to cook Chicken Dopiyaza, similarly we point out some features of birational geometry which are unique to the foliated world.

Classical Threefold	Foliated Threefold
<i>Fano Foliations</i>	

There exists smooth Fano varieties, e.g. $\mathbb{P}^n$.

Due to [1] every Fano foliation has strictly log canonical singularities (in particular Dcritical singularities).

Relation between different outcomes (rank one case)

1. Starting from three-dimensional variety X with K_X pseudo-effective, there can be more than one end products

Theorem 5(Pre-print in prep with P. Cascini, C. Spicer)

1. Let X be a normal projective variety of any dimension with klt sing and $\mathcal{F}$ be a rank one foliation with canonical singularity having $K_{\mathcal{F}}$ pseudo-effective. Then end result of foliated MMP is unique.

2. Two Sarkisov related Mori fiber spaces do not necessarily have birational bases

Theorem 6([7])

2. For threefolds endowed with rank 1 foliation with, if two foliated Mori fiber spaces coming as outcome of two different foliated MMP from same starting point, they have birational bases.

Prototype example:- $\mathbb{P}^3 \rightarrow pt$ and $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$ are end points of two different classical MMP starting from same variety but they can be never endpoints for such foliated MMPs.

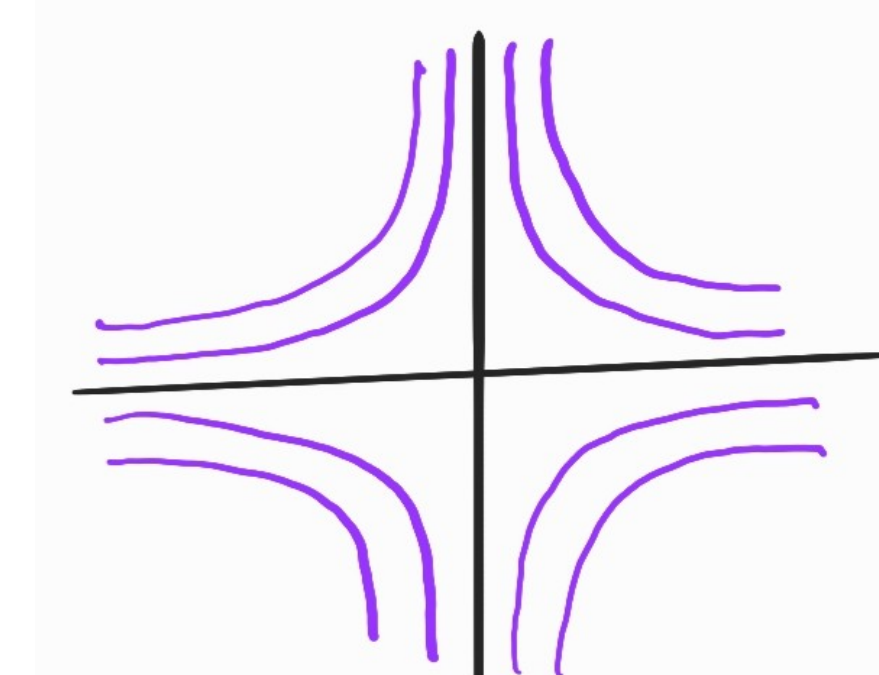


Fig. 3: $\omega = xdy + ydx$.

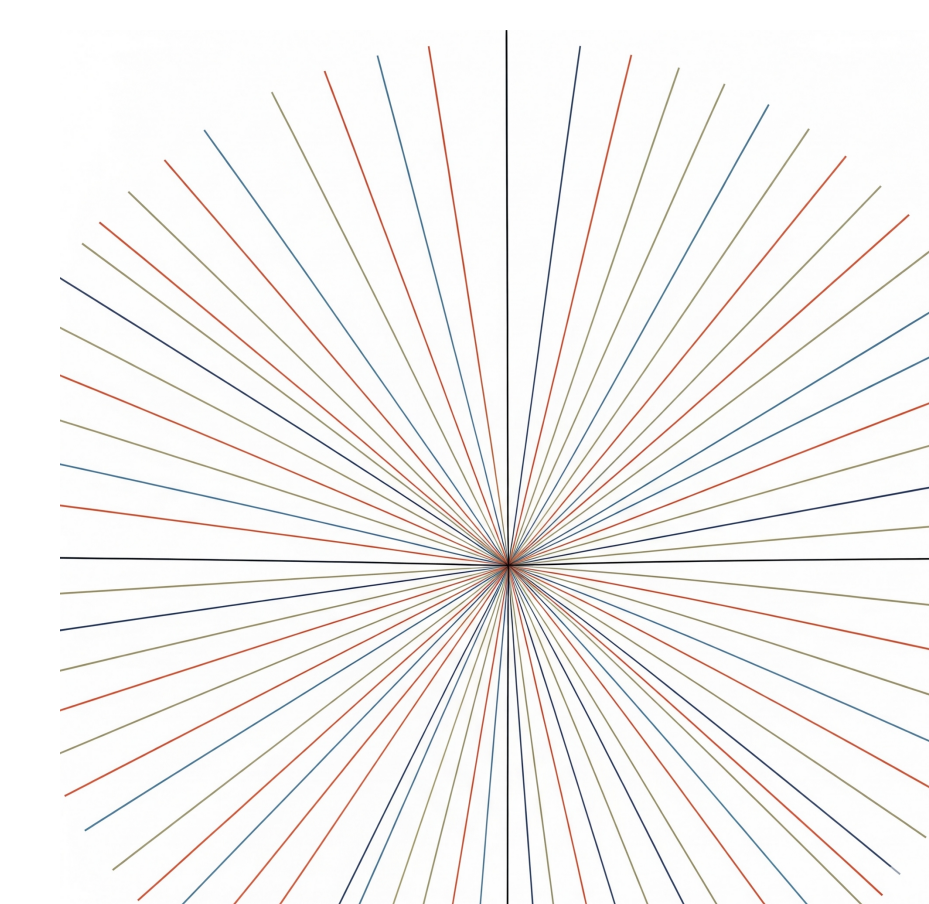


Fig. 4: $\omega = xdy - ydx$.